where R_p is the radius of individual particles; hence, allowing for void space, one obtains

$$N \simeq \frac{\pi}{6} p \left(\frac{R_s}{R_c}\right)^3 + \frac{\pi}{1.6\alpha}\left[1 - p\left(\frac{R_p}{R_c}\right)^3\right] \sum_{i=1}^{p} \left\{ \left[\frac{R_s}{R_c} - 2\cos\left\langle \frac{1}{3}\cos^{-1}\left(\frac{2i}{p} - 1\right) + \frac{\pi}{3}\right\rangle\right]^2 \cdot i \right\} \quad (20)$$

The expected values C_i and C'_p are mean values, given an infinite number of samples. Variations about these numbers will exist. Furthermore, the composition of the sample will be affected by the variability in the composition of the clusters in the sample.

Consider a single component of the mixture having an overall concentration X in the mixture. Then, the expected number of particles of the component in the sample contributed by the i-cut is

$$N_i = i \cdot C_i \cdot X$$

with variance

$$\sigma_i^2 = \sigma^2 (i \cdot C_i \cdot X) = (i \cdot C_i)^2 \sigma_{iX}^2 + (i \cdot X)^2 \sigma_{ci}^2 \quad (21)$$

The variance σ_{iX}^2 is obtained from the binomial distribution as though the mixture consisted of clusters of i particles each, and as though C_i such clusters were picked. Thus

$$\sigma_{iX}^2 = \frac{X(1-X)}{C_i} \quad (22)$$

For the whole sample, if the number N of particles in the sample is assumed invariant, the composition variance is given by the follownig expression:

$$\sigma_s^2 = \frac{\sum_{i=1}^{p} \sigma_i^2 + \sigma_u^2}{N^2} \quad (23)$$

If one substitutes Equations (21) and (22) into Equation (23), one gets

$$\sigma_s^2 = \frac{1}{N^2} \left\{ \sum_{i=1}^{p} [i^2 \cdot C_i X(1-X) + i^2 X^2 \sigma_{ci}^2] + C'_p \cdot p^2 X(1-X) + X^2 p^2 \sigma_{cp'}^2 \right\} \quad (24)$$

When N is sufficiently large that $\sigma_{cp'}^2$ is small compared with C'_p, or when X is small, Equation (24) can be simplified. Since N is constant

$$\sum_{i=1}^{p} i^2 \sigma_{ci}^2 = -p^2 \sigma c p'^2 \quad (25)$$

then, for $X^2 p^2 \sigma_{cp}^2$ negligible, one gets

$$\sigma_s^2 = \frac{X(1-X)}{N^2} \left[\sum_{i=1}^{p} C_i \cdot i^2 + C'_p \cdot p^2\right] \quad (3)$$

or, in terms of Equations (18) and (19)

$$\sigma_s^2 = \frac{X(1-X)}{N^2} \left\{ \sum_{i=1}^{p} \left[\frac{i^2\pi}{1.6\alpha}\left(\frac{R_s}{R_c} - \beta\right)^2\right] + p^2\left[\frac{\pi}{6}\left(\frac{R_s}{R_c}\right)^3 - \frac{\pi}{1.6\alpha}\left(\frac{R_p}{R_c}\right)^3 \sum_{i=1}^{p} \left(\frac{R_s}{R_c} - \beta\right)^2 i\right]\right\} \quad (26)$$

where

$$\beta = 2\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2i}{p} - 1\right) + \frac{\pi}{3}\right]$$

Equation (26) was used to calculate the values of σ_s^2/σ_R^2 used in Figure 1. As can be seen from that plot a sample size of one to ten thousand particles is an acceptable lower limit for use of Equation (2) unless mixing is very poor. Samples much smaller than that will reflect a pseudo mixing effect of sampling to a degree which may be unacceptably large.

Dynamic Optimization with Constraints from Wiener's Techniques

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Most process optimization techniques under development in recent years were obtained as fallout from military and communications applications. Interest in these methods within the chemical and process industries has been stimulated in part by competitive and other economic pressures to reduce operating costs. However, these optimization techniques usually require extensive modification before they are suited to most process industry problems.

This paper describes the development of design equations for composite feedback-feedforward process controllers based on a technique originated by Wiener (*12*) for studies in communications engineering. Controllers based on these equations were subsequently applied to various analog simulations as well as to a physical experimental laboratory system. The results of the applications are de-

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scribed elsewhere (9). The purpose of the present work is to show where the basic equations come from and how it was necessary to modify the ideal equations with many constraints in order to obtain physically meaningful results.

BASIC EQUATIONS FOR UNCONSTRAINED SYSTEMS

The use of Wiener's technique necessarily implies several assumptions.

First, it is assumed that during normal operation the system is subject to disturbances and that the objective of the control law is to compensate for the effect of these disturbances. This is different from the assumptions normally taken for application of many other methods where the plant is assumed to be initially at some perturbed state and the control objective is to return the system to the desired conditions.

Second, it is assumed that the disturbance or forcing function is a stationary random process with known statistical characteristics. That is, the average properties of the random disturbance do not change with time except for statistical fluctuations. Thus, the mean value, the variance, and other statistical parameters used to characterize this random function are assumed to be independent of the particular time of sampling.

Third, it is assumed that the controller performs a linear operation on the inputs. Thus the possibility of design of a saturating bang-bang control system is eliminated.

Fourth, the controller must be physically realizable. This means that it must be stable and that it cannot be expected to respond before the signal is received—it must be nonpredictive.

And last, the measure of merit in the selection of the optimal control law shall be the mean square of the difference between the desired and the actual outputs.

In order to obtain an explicit analytic solution to the control problem it is also necessary that the system or plant be described by time invariant linear differential equations. To facilitate this discussion, only the case of a single output variable plant will be treated in detail.

Under these conditions, the process dynamics can be described by the differential equation

$$\frac{d^n c}{dt^n} + a_{n-1}\frac{d^{n-1}c}{dt^{n-1}} + \ldots + a_o c = b_j\frac{d^i m}{dt^i} + \ldots + b_o m + g_k\frac{d^k d}{dt^k} + \ldots + g_o d \quad (1)$$

It is most convenient to work with the Laplace transformation of this equation that results from assuming zero initial conditions. The transformation of Equation (1) into the Laplace domain yields

$$C(s) = P_D(s)\ D(s) + P_M(s)\ M(s) \quad (2)$$

If dead times are absent, $P_D(s)$ and $P_M(s)$ are rational functions in the Laplace transform variable s, while if dead times are present, the rational functions are multiplied by an exponential factor in s.

It is assumed here that the ideal steady state function is a constant output, and a change of coordinates has been made so that the ideal value of the output variable is zero. Thus the design objective is to define transfer functions of a composite control system so that a minimum of the mean square value of $C(s)$ will occur when the manipulative variable is generated as

$$M(s) = Q_D(s)\ D(s) - Q_C(s)\ C(s) \quad (3)$$

The function $Q_C(s)$ is the feedback transfer function since its control action is based on information feedback from the output. $Q_D(s)$ is the feedforward transfer function since its control action is based on information from the input (Figure 1).

In order to define the control functions, the ratio of the output to the disturbance is defined in terms of a general unknown transfer function. This definition must preclude the possibility of development of a physically nonrealizable control function. Suppose that a composite unknown control function T is defined in such a way that the ratio of the output to the input is given as

$$\frac{C}{D} = T + P_D \quad (4)$$

So long as the plant P_D is stable and nonpredictive, the ratio C/D will be stable and nonpredictive if T is. Stability is an additive property. If the definition for C/D is put into the control equation, the ratio of the manipulative variable to the input becomes

$$\frac{M}{D} = \frac{T}{P_M} \quad (5)$$

Many process plants contain nonminimum phase elements such as dead times which appear in P_M as negative exponentials in s, $e^{-\tau s}$. The transfer function P_M may also contain factors in the numerator of the form $a - s$, for $a > 0$. Since P_M is in the denominator of (5), terms such as $(a - s)$ will cause the ratio M/D to be unstable. A dead time in P_M becomes a positive exponential for M/D, which means that the controller must correctly predict the random disturbance to produce proper compensation. The nonminimum phase quantities cannot be cancelled by factors in T, since real cancellation would not be perfect. A couple would be produced that would saturate the controller from the effect of extraneous noise.

Instead, in such cases, some limitation in performance must be accepted. The control equation for T is redefined as

$$\frac{C}{D} = B_M T + P_D \quad (6)$$

where B_M contains all of the nonminimum phase factors of the transfer function P_M. Then it follows that the manipulative ratio is

$$\frac{M}{D} = \frac{B_M T}{P_M} \quad (7)$$

which automatically and exactly cancels the undesirable elements from the numerator of P_M so that both ratios are realizable.

Wiener's technique is developed by finding the necessary and sufficient conditions for a minimum of the mean-square value of the output variable. The mean-square value for the output variable described in (6) is given by (1, 6, 8, 10, 12)

$$\langle c(t)^2 \rangle = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (B_M T + P_D)(\overline{B}_M \overline{T} + \overline{P}_D)\ \Phi_{DD}\ ds \quad (8)$$

The bar over the functions denotes the complement of the function

$$\overline{T} = T(-s), \quad \text{etc.}$$

The brackets $\langle \ldots \rangle$ indicate the mean value, and Φ_{DD} is the spectral density of D defined as

$$\Phi_{DD} = \int_{-\infty}^{\infty} \langle d(t)\ d(t+\tau) \rangle\ e^{-s\tau}\ d\tau \quad (9)$$

The necessary and sufficient condition defining the particular control function T which minimizes the integral in (8) is given by

$$\bar{B}_M\ (B_M T + P_D)\ \Phi_{DD} = X \tag{10}$$

where X is an arbitrary function except that it has poles with positive real parts only. That is, all of the roots of the denominator of X lie in the right half of the complex plane.

Wiener (*12*, see also *1, 6, 8, 10*) showed that the solution to (10) is given by

$$T = -\frac{1}{B_M D}\left[\frac{D\ P_D}{B_M}\right]_+ \tag{11}$$

The function D has all roots of both its numerator and its denominator in the left half of the complex plane and also satisfies the relationship

$$D\ \bar{D} = \Phi_{DD}$$

The brackets $[\ldots]_+$ indicate that only the physically realizable (stable and nonpredictive) portion of the function has been retained. This portion corresponds to the residues of the function at left-half plane poles.

Consider this solution where the plant is minimum phase; that is, where $B_M = 1$. It follows directly that

$$T = -P_D \tag{12}$$

Substitution of this result into (2), (3), and (6) yields

$$\frac{Q_D - P_D Q_C}{1 + P_M Q_C} = -\frac{P_D}{P_M} \tag{13}$$

Since both Q_D and Q_C may be chosen, it can be seen that there are an infinite set of control laws which may yield this net result. It is of interest, however, to consider several special cases. First, the system without feedback, that is, when $Q_C = 0$. Then it follows that

$$Q_D = -\frac{P_D}{P_M} \tag{14}$$

This relationship is the ideal or nominal feedforward control law such as would be specified by ordinary feedforward design equations. It produces an output that is theoretically identically equal to zero. In other words, optimal control of an unconstrained minimum phase plant means zero output regardless of the input load.

In the absence of feedforward, that is for $Q_D = 0$, it follows that

$$\frac{-P_D Q_C}{1 + P_M Q_C} = -\frac{P_D}{P_M} \tag{15}$$

This equation can be satisfied only if Q_C is infinitely large. Thus, in a theoretically perfect system, feedforward can be matched only by infinite feedback control.

In order to treat a case containing nonminimum phase elements, it is necessary to assume a specific functional form for the process plant. In the work being described (*8*), general design equations were developed for third-order plants with dead times. For purposes of this presentation, the model has been restricted to first order with dead times. There is no essential difference in using the two different models—just a moderate reduction of the complicated algebra that is involved.

Thus, assume that the plant can be described by the first-order differential equation. Hence

$$C(s) = \frac{K_D}{\alpha + s}\, e^{-\tau_c s}\ D(s) + \frac{K_M}{\alpha + s}\, e^{-(\tau_c + \tau_M)s}\ M(s) \tag{16}$$

Note that a different gain and additional dead time are the only dynamic elements assigned to the control system.

In order to solve the design equations, it is necessary also to assume a form for the spectral density of the disturbance. This was taken as

$$\Phi_{DD}(s) = \frac{\mu^2}{\sigma^2 - s^2} \tag{17}$$

This equation is an exact form representing constant magnitude square waves whose signs change as a random function of time, square waves with randomly changing signs and amplitude, and a Gaussian noise produced by passing white noise through a first-order filter. In general, signals with identical spectral density may have quite different time behavior.

With (16) and (17) used, (10) becomes

$$\frac{T\mu}{\sigma + s} + \frac{K_D e^{\tau_M s}}{\alpha + s} \cdot \frac{\mu}{\sigma + s} = \frac{X}{D} \tag{18}$$

The second term on the left-hand side of (18) must be split into its physically realizable and unrealizable parts. Although this term has only left half plane (l.h.p.) poles, the exponential factor in the numerator makes part of it nonzero for $t < 0$. To see this point, consider a general case of a sum of fractions of the type $(ke^{\tau s})/(\beta + s)$, where $\beta, \tau > 0$. (It is assumed throughout that the roots of the denominator are *distinct*. If they are not, the method of finding residues is different, but the results are the same.) The inverse transform of this factor is $ke^{-\beta(t+\tau)}\ u(t + \tau)$, where $u(t)$ is the unit step function. A plot of this result is shown in Figure 2. The control function T cannot compensate for error which occurs for $t < 0$; hence the only part of Figure 2 that is useful for design purposes lies to the right of $t = 0$. This function is described by $ke^{-\beta(t+\tau)}u(t)$ which has the transform of $(ke^{-\beta\tau})/(\beta + s)$. This is the principal part of the original term at the l.h.p. pole.

Solution now proceeds by retaining only the principal parts of the second term at l.h.p. poles which gives

$$\frac{T\mu}{\sigma + s} + \frac{K_D\mu}{\sigma - \alpha}\left[\frac{e^{-\tau_M\alpha}}{\alpha + s} - \frac{e^{-\tau_M\sigma}}{\sigma + s}\right] = 0 \tag{19}$$

From (19) it follows that

$$T = -\frac{P_D}{B_M}\ (T_0 + T_1 s) \tag{20}$$

where

$$T_0 \stackrel{\Delta}{=} \frac{\alpha e^{-\tau_M\sigma} - \sigma e^{-\tau_M\alpha}}{\alpha - \sigma}$$

and

$$T_1 \stackrel{\Delta}{=} \frac{e^{-\tau_M\sigma} - e^{-\tau_M\alpha}}{\alpha - \sigma}$$

The relation between the specific controller functions Q_C and Q_D is found by substitution of (20) into (2), (3), and (6):

$$-P_D\ (T_0 + T_1 s) = \frac{P_M}{B_M}\ \frac{Q_D - P_D Q_C}{1 + P_M Q_C} \tag{21}$$

Again consider the controller for no feedback, that is, $Q_C = 0$, so that

$$Q_D = -\frac{B_M P_D}{P_M}\ (T_0 + T_1 s) \tag{22}$$

This result is the classical solution to the predictor problem. If a control system is asked to anticipate a random signal, the control action is attenuated to compensate only for that part of the response it knows about. Differentiation of the signal is also used to attempt to predict trends based on average frequencies.

The situation for feedback control only is found by allowing the feedforward function Q_D to be zero. Then from (21)

$$Q_C = \frac{e^{\tau_C s}\,(T_0 + T_1 s)}{\dfrac{P_M}{B_M}\,[1 - e^{-\tau_M s}\,(T_0 + T_1 s)]} \tag{23}$$

This solution is valid, and Q_C is physically realizable if $\tau_C = 0$. Thus a feedback controller has been found which controls with the same degree of effectiveness as feedforward control. This equivalence is theoretically possible if there is a finite dead time in the controller but not in the output sensing loop, so that both feedback and feedforward are subject to the same dead time.

This result can be clarified by consideration of the physical situation. If there is no dead time in the controller, the output response can theoretically be reduced to zero by feedforward control. Finite feedback control cannot do this, since the control action cannot be keyed to a variable that is identically zero. If pure transport delays exist, however, a zero output cannot be achieved even by a perfect feedforward controller. In such a case, there is a finite output to which a feedback controller can be keyed. All that is necessary is to determine the functionality relating input to output and then simply to design the feedback controller to give the same control signal as that of the feedforward controller.

Control System with Constraint on Control Effort

In actual practice there are constraints which prevent the realization of the control as specified by the above equations. In particular, constraints may be imposed on control effort.

The control effort may be constrained in a number of ways that can be implemented within the framework of Wiener's method. For example, the manipulative variable may be constrained as to the maximum rate of change as, for example, of steam pressure in a calandria. However, for the examples considered here, constraints will be taken only on the maximum value attainable for the controlling variable.

The incorporation of constraints into the design equations is accomplished by the use of Lagrange multipliers. Instead of minimizing the mean-square output, an objective function is formed from the weighted sum of the mean-square output and the quantities to be constrained, in this case, the mean-square value of the control effort:

$$F(T_1\lambda) = \langle c(t)^2\rangle + \lambda^2 \langle m(t)^2\rangle \tag{24}$$

The mean-square value of the control effort is computed by using (7) as

$$\langle m(t)^2\rangle = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{T\,\overline{T}}{P_M \overline{P}_M}\,\Phi_{DD}\,ds \leq \mathcal{L} \tag{25}$$

Since (6) and (7) give expressions for C/D and M/D, it's possible to describe the sum (24) in terms of the functions already considered. Then direct application of Wiener's method yields an overall control law T as a function of the weighting factor λ.

This function can be substituted into (25) and by changing the inequality to an equality, a numerical value for the weighting factor λ can be found. Since the sum has been minimized, and since the constraint was made to be obeyed exactly, the first term of (224) is minimized if the relation of C to M is montonic. It is inconceivable that optimal use of more control effort could result in a deterioration of control. Thus, it can be inferred that the given λ minimizes the output for the given control effort constraint.

Application of Wiener's technique to (24) given an equation equivalent to (10):

$$\left[\overline{B}_M(B_M T + P_D) + \frac{\lambda^2 T}{P_M \overline{P}_M}\right]\Phi_{DD} = X \tag{26}$$

Rearranging and using the definition

$$1 + \frac{\lambda^2}{P_M \overline{P}_M} = Y\overline{Y} \tag{27}$$

one gets

$$T\,Y\,D + \frac{P_D D\,\overline{B}_M}{\overline{Y}} = \frac{X}{\overline{D}\overline{Y}} \tag{28}$$

From this it follows that

$$T = -\frac{1}{YD}\left[\frac{P_D D\,\overline{B}_M}{\overline{Y}}\right]_+ \tag{29}$$

To illustrate this general discussion, the specific solution will be obtained for the system described by (19). Thus (27) becomes

$$Y\overline{Y} = 1 + \frac{\lambda^2}{K_M^2}(\alpha^2 - s^2) \tag{30}$$

and from (29) it follows that

$$T = \frac{-K_M^2}{\lambda^2}\,\frac{\sigma + s}{\beta + s}\left(\frac{K_D e^{\tau_M s}}{(\sigma + s)(\alpha + s)(\beta - s)}\right)_+ \tag{31}$$

where

$$\beta^2 = \alpha^2 + \frac{K_M^2}{\lambda^2}$$

After the principal parts of the bracketed expression have been found at l.h.p. poles, (31) becomes

$$T = -\frac{P_D}{B_M}\,\frac{T_0 + T_1 s}{\beta + s} \tag{32}$$

where

$$T_0 = \frac{K_M^2}{\lambda^2(\sigma - \alpha)}\left[\frac{\sigma e^{-\tau_M \alpha}}{\beta + \alpha} - \frac{\alpha e^{-\tau_M \sigma}}{\beta + \sigma}\right]$$

$$T_1 = \frac{K_M^2}{\lambda^2(\sigma - \alpha)}\left[\frac{e^{-\tau_M \alpha}}{\beta + \alpha} - \frac{e^{-\tau_M \sigma}}{\beta + \sigma}\right]$$

Note that the only unknown on the right-hand side of (32) is λ which becomes fixed by (25).

Equation (32) can give specific expressions for Q_D and Q_C when one of the two are fixed. Letting $Q_C = 0$ gives

$$Q_D = -\frac{K_D}{K_M}\left(\frac{T_0 + T_1 s}{\beta + s}\right) \tag{33}$$

When $Q_D = 0$, (32) yields

$$Q_C = \frac{\left(\dfrac{T_D + T_1 s}{K_M}\right)\left(\dfrac{\alpha + s}{\beta + s}\right)\cdot e^{\tau_C s}}{1 + \left(\dfrac{T_0 + T_1 s}{\beta + s}\right)e^{-\tau_M s}} \tag{34}$$

It is noted here again that if the feedback dead time τ_C is zero, it is possible to define a physically realizable feedback controller that theoretically performs as effectively as feedforward in output attenuation. As before, the explanation lies in the fact that perfect output attenuation is not expected of the feedforward controller so that finite output exists to which feedback can be keyed. In (34) as in (23) the numerator of the feedback controller is 1 deg. higher in s (or differentiation) than the equivalent feedforward controller. However, the feedforward controller operates directly on the input disturbance, while the feedback controller operates on the partially integrated and smoothed output. Intuitively the net result can be seen to be equivalent.

Thus, in the absence of constraints and nonminimum phase elements, the ideal overall control function was

$$T = -\frac{P_D}{B_M}$$

When dead times were considered, the control function became a partial differentiator:

$$T = -\frac{P_D}{B_M}(T_0 + T_1 s)$$

With an added constraint on control effort, the controller has now become

$$T = -\frac{P_D}{B_M}\frac{T_0 + T_1 s}{\beta + s}$$

The predictiveness of this function is tempered by the control effort constraint.

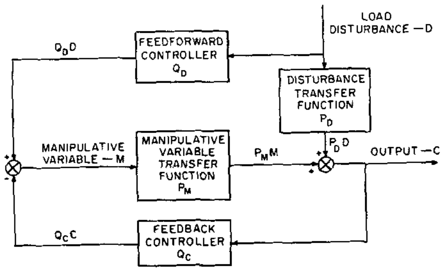

Fig. 1. Block diagram of plants and controllers under consideration.

Effect of Error in the Mathematical Model

In the previous development, optimization goals have been achieved by using only one of the two available degrees of freedom. In a particular application, circumstances may make it necessary to choose one control function to be zero. In general, however, the best combination of Q_C and Q_D will be sought. Additional constraints will be required to specify the individual functions Q_C and Q_D uniquely. One additional constraint will be defined through examination of feedforward control limitations. These limitations have made it secondary in use to feedback, even though the previous equations seem to indicate the equality if not the superiority of feedforward.

One of the principal weaknesses of feedforward control is its sensitivity to error in the mathematical model used in its design. If feedforward control is based on an inaccurate model, it will produce partially inappropriate control action. No knowledge of its error is transmitted back to the controller, since feedforward compensation is based only on the value of the input.

Model error can be divided into two classes: permanent error, that is, a value of a parameter which has been assumed or measured inaccurately; or transient error, that is, parameter values which change with time. The former type of error may succumb to development of accurate model identification techniques or can be tuned out by adaptive control practices. However the variability of transient error will cause model error output indefinitely. Examples of the latter range from scaling of heat transfer surfaces and ambient temperature cycling to variations of catalyst concentration or activity. In short, this type of error is a form of unmeasured disturbance which is regraded here in relation to the transfer functions of the monitored variables. An optimal controller must take this type of error into consideration.

Consideration of the model error was incorporated into the optimal design equations by computing a model error output as defined by

$$\Delta C = (\Delta B_M \Delta T + \Delta P_D) D \tag{35}$$

The Δ terms are incremental quantities resulting from misidentification. These quantities can be any error not correlated with inputs. The effect of small system nonlinearities can be lumped into this term. However, the disturbance acting through the nonlinearities is likely to produce partially correlatable output.

Explicit Determination of Parameter Error

If the output defined by (35) is to be included in the optimization procedure, the Δ's must be related to the primary parameter variations. Assume that the various functions can be expanded into Maclaurin's series with respect to the parameter variations and that

$$\Delta P = P(\xi) - P(0) = \left.\frac{\partial P}{\partial \xi}\right|_{\xi=0} \xi \tag{36}$$

The term $\partial P/\partial \xi$ is the error coefficient (*11*).

In all of the problems considered here, the operators are rational functions in s multiplied by an exponential time lag. Therefore representation of the general transfer function with its errors is

$$P(\xi) = (K + \Delta K) \cdot \frac{\pi_i (z_i + \Delta z_i + s)}{\pi_j (p_j + \Delta p_j + s)} e^{-(E+\Delta E)s} \tag{37}$$

where ξ is the vector $[\Delta K, \Delta z_1, \ldots, \Delta z_n, \Delta p_m, \Delta E]$.

From (36) and (37) the error due to parameter variation may be approximated as

$$\Delta P = \left[\frac{\Delta K}{K} + \sum_i \frac{\Delta z_i}{z_i + s} - \sum_j \frac{\Delta p_j}{p_j + s} - \Delta E_s s \right] P \tag{38}$$

The function T is affected by errors in the plant functions as well as errors in the controller functions. Normally, however, any variations in the controller will be at least an order of magnitude less than those in the plant functions. Thus, it is assumed that

$$\Delta Q_C = \Delta Q_D = 0 \tag{39}$$

Therefore

$$\Delta T = \frac{\partial T}{\partial P_M}\Delta P_M + \frac{\partial T}{\partial P_D}\Delta P_D \tag{40}$$

Evaluation of the partial derivatives gives

$$\Delta T = \frac{T_D}{1 + P_M Q_C}\frac{\Delta P_M}{P_M} - \frac{P_D P_M Q_C}{1 + P_M Q_C}\frac{\Delta P_D}{P_D} \tag{41}$$

The above equations introduce a quantity that cannot be computed in terms of simple input-output terms. Hence it becomes necessary to define a second control function in addition to the input-output ratio T. The most convenient form of this new control function is in terms of feedback control only:

$$L = \frac{P_M Q_C}{B_M(1 + P_M Q_C)} \tag{42}$$

In this case, the model error output from (35) becomes

$$\Delta C = \left(T \frac{B_M}{P_M} \Delta P_M + \Delta P_D \right) (1 - B_M L) D \quad (43)$$

From this equation, the mean-square model error output can be computed and a new Lagrange multiplier sum formed:

$$F = <\Delta c^2> + \lambda_c^2 <c^2> + \lambda_M^2 <m^2> \quad (44)$$

Equation (44) is just one of six equivalent permutations that could be used since equalities have been substituted for inequalities. This particular form implies minimization of the model error output for given levels of nominal mean-square error and mean-square control effort.

Application of Wiener's method to (44) generates two equations containing the two unknown control functions:

$$\overline{B}_M \frac{\Delta \overline{P}_M}{\overline{P}_M} \left(B_M T \frac{\Delta P_M}{P_M} + \Delta P_D \right) (1 - B_M L) \cdot$$

$$(1 - \overline{B_M L}) \Phi_{DD} + \lambda_c^2 \overline{B}_M (B_M T + P_D) D \overline{D}$$

$$+ \lambda_M^2 \frac{T}{P_M P_M} \Phi_{DD} = X_1 \quad (45)$$

and

$$\frac{\overline{B}_M \overline{B}_D}{\overline{P}_D} (1 - B_M L) \left(B_M T \frac{\Delta P_M}{P_M} + \Delta P_D \right) \cdot$$

$$\left(\overline{B_M} T \frac{\overline{\Delta P_M}}{\overline{P_M}} + \overline{\Delta P_D} \right) \Phi_{DD} = X_2 \quad (46)$$

The functions X_1 and X_2 have right half plane (r.h.p. poles only.

Solution of (46) proceeds by noting that the last three factors form a symmetric function which is factorable into parts with l.h.p. poles and zeros and r.h.p. poles and zeros. Let these factors be Y and $\overline{Y}$, respectively. Then

$$\frac{\overline{B_M B_D}}{P_D} (1 - B_M L) Y = \frac{X_2}{\overline{Y}} \quad (47)$$

If the transfer functions are all minimum phase, that is, if B_M and B_D are both unity, (47) can be satisfied if $L = 1$ or if $Y = 0$. If $Y = 0$, the first term of (45) is also zero, and it would be nothing more than coincidence if T were defined so that it also satisfied the second part of (45). Therefore, the solution $L = 1$ will be examined. The more complicated problem of nonminimum phase transfer functions would require evaluation of physically realizable parts.

From (42), $L = 1$ if and only if Q_C approaches infinity at all frequencies, a solution which has been met earlier. Here, however, the situation is different; Q_D has not been arbitrarily set equal to zero indeed, there is no arbitrariness whatsoever in this solution.

It can be shown that

$$Q_D = \frac{1}{P_M} (T + P_D) \quad (48)$$

Thus, although Q_C approaches infinity, Q_D remains a finite function. It balances the overall control function T so that control effort and output attenuation constraints are met. This result contradicts the notion that is sometimes implied that infinite gain feedback theoretically causes infinite control effort. The infinite gain feedback in this computation achieves near zero output by adding mirror images of both the measured and unmeasured disturbance to the plant. Of course this feedback is physically unrealizable despite the fact that it is within previous constraints, so that a further examination of design bases and constraints is in order.

The result which pits an infinite feedback system against the feedforward to achieve a finite, optimal overall transfer function derives, in part, from assuming zero error in the controller. These results also stem from the fact that continuous equations are describing a physical phenomena which has, in fact, a limited threshold.

The feedback controller in the above design is being called upon to make corrections based upon very small output changes. Even if a zero mean-square output is not required, the feedback controller attempts to maintain the variance of the output as near to the constraint limit as possible to minimize other values. To achieve this result, maximum feedback gain also is employed.

Design Equations for Finite Feedback Control

In control systems of the type considered here, the controller measures the value of some state variable and then computes a control effort that depends on the value of this state variable. In a real situation, it processes not only the value of the given state variable but also any error that is present in the sensed value of the variable and any noise generated in the controller. The system output is therefore not the result of only the primary disturbance and the ideal computed control effort, but in addition it is corrupted by control effort erroneously computed from controller noise. This excess control effort also contributes to saturation of the controller capacity. These two effects help limit the degree of control effort that

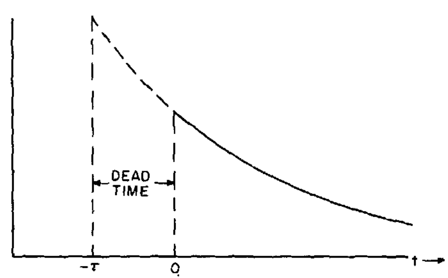

Fig. 2. Inverse transform of $\frac{ke^{Ts}}{\beta + s}$.

can be keyed to a given variable. The signal-to-noise ratio of the quantity which ultimately activates the manipulative variable must be sufficiently high so that there is more disturbance attenuation than output corruption.

These effects are usually much more important in the feedback portion of the control system than the feedforward. Disturbance signals important enough to affect a plant output significantly are generally large enough to give high signal-to-noise ratios in a controller. On the other hand, feedback controllers are particularly susceptible to signal-to-noise ratio difficulties. The very success of any control effort adds to the problem; as the output becomes closer to the ideal of zero, the signal-to-noise ratio of the feedback diminishes. In the limit, the feedback controller will be attempting to compensate for signals generated within itself and can tend to produce more output than it eliminates. For this reason, a signal-to-noise type of constraint is imposed only on the feedback system.

The formulation of a gain constraint on the feedback presents somewhat of a problem if it is to be in a form amenable to solution by Wiener's techniques. It would also be desirable to avoid introduction of another Lagrange multiplier, since each new variable introduces another dimension.

The method to be used here will be to alter (3) so that the feedback function Q_C operates not only on the output C but also on a random noise factor δ. (The same Greek letter will be used for the factor and its transform; the context should indicate which is meant if the argument is not stated.) The factor δ tends to mask the effect of small outputs so that most feedback control effort is delayed until the output exceeds δ. Thus the response somewhat resembles that of a system with a small dead time. The manipulative variable then becomes

$$M = Q_D D - Q_C(C + \delta) \qquad (48)$$

The other equations remain unchanged.

Combining (48) with (2) and (6) one gets

$$M = \frac{B_M}{P_M}(T\ D - L\ \delta) \qquad (49)$$

and

$$C = (B_M T + P_D)D - B_M\ L\ \delta \qquad (50)$$

Use of these equations with (35) gives

$$\Delta C = \left(\frac{B_M T}{P_M} \Delta P_M + \Delta P_D \right) (1 - B_M L) D \qquad (51)$$

The term δ does not appear in (2) and hence is introduced into (50) by (48). It is neglected in finding ΔC.

These equations are solved as before. The necessary and sufficient conditions to minimize the sum (44) are

$$\overline{B}_M \frac{\overline{\Delta P_M}}{\overline{P_M}} \left(B_M T \frac{\Delta P_M}{P_M} + \Delta P_D \right) (1 - \overline{B}_M L)(1 - \overline{B_M L})\ \Phi_{DD} + \lambda_C^2\ \overline{B_M}\ (B_M\ T + P_D)\Phi_{DD}\ \lambda_M^2\ B_M \overline{B_M} \frac{T}{P_M \overline{P_M}} \Phi_{DD} = X_1 \qquad (52)$$

$$\frac{\overline{B_M B_D}}{P_D} \left(B_M T \frac{\Delta P_M}{P_M} + \Delta P_D \right) \left(\overline{B_M T \frac{\Delta P_M}{P_M} + \Delta P_D} \right) (1 - B_M L)\Phi_{DD} + \lambda_C^2\ B_M \overline{B_M}\ L\ \Phi_{\delta\delta} + \lambda_M^2\ B_M \overline{B_M}\ \frac{L}{P_M \overline{P_M}}\ \Phi_{\delta\delta} = X_2 \qquad (53)$$

The spectral density of δ appears only in the second of these equations which, as will be seen, determines L. The noise factor δ affects the overall transfer function only insofar as it affects L, thereby changing the amount of identification uncertainty that must be allowed for in the feedforward control.

Since there are two equations with two unknowns, in principle it is possible to reduce by elimination to one equation. Since the unknowns are functions, however, and functions with differing arguments at that, this elementary approach will not work. Therefore, recourse must be taken to successive numerical approximations to obtain a solution.

The general method of solution is based on the idea that the principal reason for existence of the control system is the elimination of output, that is, to make $C \equiv 0$. It was seen previously that a successful design for $C \doteq 0$ is $B_M T \doteq -P_D$. On the other hand, if the control effort is very small because of constraints, then $B_M T \doteq 0$. Although a number of other important factors have been introduced to the equations, an approximation to the overall control effort would be some constant γ times P_D; that is, let

$$B_M T \doteq -\gamma P_D, \quad 0 \leqq \gamma \leqq 1 \qquad (54)$$

Equations (52), (53), and (54) are then solved by successive approximations adjusting the value for γ until

$$\langle c(t)^2 \rangle_{\text{controlled}} \doteq (1 - \gamma)^2 \langle c(t)^2 \rangle_{\text{uncontrolled}} \qquad (55)$$

The resulting control functions would then be very near optimal.

Actual calculations (8) indicate that the control design is quite insensitive to the value used for γ in (54). This result is logical, since often the error factors themselves are little more than shrewd guesses and seldom are functions known with a high degree of accuracy. A choice of $\gamma = 1$ was found to be quite satisfactory for reasonably effective controllers.

The explicit form of the overall functions resulting from these equations are, of course, quite complicated. For a first-order plant, the feedback function L was a linear factor divided by a cubic, and the overall function T was a sixth-order polynomial divided by a ninth order.

The explicit control functions Q_C and Q_D were, of course, even more involved and must include exponential dead times δ. The feedback function is given by

$$Q_C = \frac{L}{P_M(1 - L\ e^{-\tau_M s})} \qquad (56)$$

The feedforward control function is found by essentially subtracting the feedback from the overall function:

$$Q_D = \frac{T + L\ e^{-(\tau_M + \tau_C)s}}{P_M(1 - L\ e^{-\tau_M s})} \qquad (57)$$

Of course, the parameters in these forms are variable, and the control laws could be made to take on almost any characteristics depending on the emphasis that was put on any particular constraint or variable.

SUMMARY AND CONCLUSIONS

The necessary condition for optimal composite control of a system energized by a random input was found by direct application of calculus of variations. This condition, which was the vanishing of the Wiener-Hopf integral, was shown also to be sufficient for optimality. The general explicit solution to the integral equation was found with the aid of complex variable theory.

The controller for minimization of mean-square output was found to be unrealistic because constraints normally encountered in process industry control had not been considered. As a first step to develop a general control law algorithm, a constraint on control effort was included in the design equations by means of the Lagrange multiplier. This led to solutions specifying only feedforward control.

Feedforward control does not enjoy widespread use because of its sensitivity to model inaccuracies. When minimization was performed on a weighted sum of nominal system output and system output due to expected model error, infinite feedback control was specified.

Finally a factor reflecting the signal-to-noise ratio in the control signals was incorporated into the design equations. This then led to development of valid optimal control laws.

However, even for the solution of this relatively simple case for a single input-output linear system, successive numerical approximations were required. Furthermore, solution of these equations, while in principle straightforward and explicit, actually becomes very arduous algebraically. For example, if it is assumed that the plant transfer functions P_D and P_M have at most three poles and one zero, then solution involves finding the roots of a twelfth-order polynomial (8). It is not uncommon for these polynomials to be ill conditioned so that the root finding task can be difficult.

There is another drawback to the explicit solution method described here that is not immediately evident. Use of Lagrange multiplier techniques in effect define a control system such that a weighted sum of the output variances are minimized. Thus a constraint on control effort, for example, is treated by designing a control which does not often call for more than the allotted control effort. This achievement is at the expense of less precise control at all times including that fraction of the time when saturation of the controller would not occur even for perfect control. Another control law which saturates the controller for large disturbances may provide superior overall controlled output attenuation. While the true bang-bang controller is not optimal for this set of conditions, a partial bang-bang system is superior to the nonsaturating controller provided that saturation does not imply other undesirable effects.

It is suggested that this problem be attacked by direct numerical application of constraints as true inequalities rather than the equalities of the Lagrange multiplier technique.

For multidimensional systems, the control variables in the previous treatment expand to vectors; the transfer functions becomes matrices. Several methods have been proposed to factor these spectral matrices (*2*, *13*), but it is obvious that severe algebraic complications are encountered even for a rather simplified multidimensional case with few constraints.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant Number GK 98. R. H. Luecke was the recipient of a National Science Foundation Cooperative Fellowship during a major portion of this work. The authors are indebted to the staff of the Osage Computer at the University of Oklahoma for their generous contribution of computer time and programming assistance.

NOTATION

a_i = constant
B_D = nonminimum phase portion of disturbance transfer function
B_M = nonminimum phase portion of manipulative variable transfer function
b_j = constant in system describing equation
C, $C(s)$ = Laplace transform of $c(t)$
$c(t)$ = output or controlled variable
D, $D(s)$ = Laplace transform of $d(t)$
D = minimum phase portion of spectral density of random disturbance
$d(t)$ = disturbance or load variable
E = dead time in plant transfer function
$F(A, \lambda)$ = sum of integrals to be minimized
g_i = constant in system describing equation
j = imaginary number $\sqrt{-1}$
K = gain factor of general plant transfer function
K_D = gain factor for disturbance transfer function
K_M = gain factor for manipulative variable transfer function
L, $L(s)$ = feedback control function
$\mathcal{L}$ = maximum value of mean-square control effort
M, $M(s)$ = Laplace transform of $m(t)$
$m(t)$ = manipulative or control variable
P = general plant transfer function
$P_D(s)$ = transfer function of disturbance in system Laplace domain equation
$P_M(s)$ = transfer function of manipulative variable in system Laplace domain equation
p_i = pole of plant transfer function
$Q_C(s)$ = transfer function of feedback controller
$Q_D(s)$ = transfer function of feedforward controller
s = Laplace transform variable
T, $T(s)$ = overall control function
T_n = coefficients in overall transfer function
t = time
X, X_1, X_2 = general transfer functions having poles in the r.h.p. only
Y = function that is equal to a given transfer function except that r.h.p. zeros are replaced by l.h.p. zeros of the same magnitude and exponential factors are absent
z_i = zero of plant transfer function

Greek Letters

α = system natural frequency or pole of transfer function
β = factor defined
γ = weighting factor
$\Delta\alpha$ = error in system natural frequency
ΔC = model error output or increment in output due to parameter variation
ΔE = error in exponential dead time of general plant transfer function
ΔK = error in gain of general plant transfer function
ΔP_D = error in disturbance transfer function
ΔP_M = error in manipulative variable transfer function
Δp_i = error in i^{th} pole of general plant transfer function
Δz_i = error in i^{th} zero of general plant transfer function
δ = random noise in feedback circuit
λ, λ_M, λ_C = Lagrange multiplier or weighting factors
μ = magnitude factor of disturbance
σ = mean frequency of disturbance
τ = time difference $(t_1 - t_2)$
τ_C = dead time in output circuit
τ_M = dead time in controller
$\Phi_{AB}(s)$ = cross-spectral density of random time functions, $a(t)$ and $b(t)$
$\varphi_{AB}(t_1, t_2)$ = correlation function of random functions $a(t)$ and $b(t)$
$\langle \ldots \rangle$ = mean value

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Manuscript received December 21, 1966; revision received May 15, 1967; paper accepted May 17, 1967. Paper presented at AIChE Houston meeting.